



On the Exponential Diophantine Equation $x^2 + p = 2^n$ for Primes $p \equiv 7(\text{mod}8)$

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International.



ABSTRACT

We revisit the Ramanujan–Nagell type exponential Diophantine equation $x^2 + p = 2^n$, $x, n \in \mathbb{Z} \geq 1$, p an odd prime, with emphasis on the congruence class $p \equiv 7(\text{mod}8)$. Using elementary modular arguments, we show that for $n \geq 3$ the congruence $p \equiv 7(\text{mod}8)$ is a necessary condition for solvability, but not sufficient. We then propose a reproducible search methodology (over bounded exponents and odd squares) and compile a corrected set of illustrative solutions for several primes $p \equiv 7(\text{mod}8)$ (including multiple-solution phenomena, e.g. $p = 23$). Finally, we establish a clean reduction showing that the variant $x^2 + 7 = 4^m$ has a unique positive solution $(x, m) = (3, 2)$, by linking it to the classical Ramanujan–Nagell equation

INTRODUCTION

Exponential Diophantine equations are central to the field of number theory, where they often appear as a natural generalization of classical Diophantine problems. These equations typically involve integer solutions to equations that contain exponential terms. One of the classical examples is the **Ramanujan–Nagell equation**, which takes the form:

$$x^2 + 7 = 2^n, \quad x, n \in \mathbb{Z} \geq 1,$$

Where x and n are positive integers. This equation was famously conjectured by the Indian mathematician **Srinivasa Ramanujan** in 1913 and later proven by **Tord G. Nagell** in 1948. Nagell's proof demonstrated that the only solutions to the equation are

$$(n, x) \in \{(3, 1), (4, 3), (5, 5), (7, 11), (15, 181)\}.$$

The success of this equation as a model for exploring exponential Diophantine equations inspired many mathematicians to generalize it. A natural extension of the Ramanujan–Nagell equation involves replacing the constant 7 with a general odd prime p , resulting in the equation:

$$x^2 + p = 2^n \tag{1}$$

$$x, n \in \mathbb{Z} \geq 1, p \text{ is an odd prime.}$$

The general question that arises is: for which primes p can the equation $x^2 + p = 2^n$ have integer solutions for x and n ? In this paper, we revisit the generalized Ramanujan–Nagell type exponential Diophantine equation and focus on the case where the prime p satisfies the $p \equiv 7 \pmod{8}$. This particular congruence condition plays a significant role in ensuring that integer solutions can exist for the equation when $n \geq 3$. Our work is motivated by the fact that $p \equiv 7 \pmod{8}$ emerges as a necessary condition for the existence of solutions in the general case for larger values of n .

We aim to provide a systematic exploration of this condition by using elementary modular arguments. This leads to the following result: for $n \geq 3$, the congruence $p \equiv 7 \pmod{8}$ is necessary but not sufficient for the solvability of the equation $x^2 + p = 2^n$. In other words, although primes of the form $p \equiv 7 \pmod{8}$ can appear in solutions, the solvability of the equation also depends on other more subtle arithmetic properties.

In this paper, we develop a transparent and reproducible search methodology that allows us to identify solutions to the equation for various primes $p \equiv 7 \pmod{8}$. Our search methodology is based on constructing solutions by examining bounded exponents and odd squares. We present a carefully organized list of illustrative solutions for specific primes, correcting and refining earlier work in the field. Through this approach, we uncover a variety of interesting phenomena, including instances where multiple solutions exist for a given prime p , such as with $p = 23$.

Additionally, we examine a variant of the original equation, specifically the equation $x^2 + 7 = 4m$, and show that it has a unique solution: $(x, m) = (3, 2)$. We link this variant to the classical Ramanujan–Nagell equation, providing a cleaner reduction to an already well-known result.

Thus, the primary goals of this paper are:

1. To clearly establish the necessary condition $p \equiv 7 \pmod{8}$ for the solvability of the equation $x^2 + p = 2^n$ when $n \geq 3$.
2. To propose a straightforward search method that generates integer solutions for various primes $p \equiv 7 \pmod{8}$, offering a corrected set of examples.
3. To explore the unique solution to the equation $x^2 + 7 = 4m$, linking this problem to the classical Ramanujan–Nagell equation and confirming the uniqueness of the solution.

Finally, we conclude with several recommendations for future research, which include proving completeness within bounded ranges for n and p , further examining the number of solutions for each prime $p \equiv 7 \pmod{8}$, and connecting our findings to existing general frameworks in number theory.

By revisiting and refining earlier examples, our aim is to contribute to the broader understanding of exponential Diophantine equations and their connection to prime congruences and modular arithmetic, continuing the legacy of the classical Ramanujan–Nagell equation.

LITERATURE RIVIEW

Equations of the Form

$$x^2 + D = AB^n$$

For fixed integers A, B, D are commonly described as Ramanujan–Nagell type problems; general finiteness principles (e.g., Siegel-type results) imply that for fixed parameters, the number of integer solutions is finite. Beyond the classical case $D = 7, A = 1, B = 2$, there is a substantial literature studying generalizations and bounds on the number of solutions, often combining transcendence methods, descent, and computational techniques. A representative line of work treats prime-power variants such as $x^2 + D = p^z$. Recent articles also study mixed-exponent forms such as $x^2 = 2^m + p^n$. Survey-level overviews of generalized Lebesgue–Ramanujan–Nagell equations summarize modern techniques and open problems. In this context, the present paper intentionally remains **elementary**: we focus on modular constraints and on systematically organizing example data for (1) when $p \equiv 7 \pmod{8}$

METHODOLOGY

Elementary Modular Filtering

Assume p is an odd prime and (x, n) satisfies (1) :

- If x is even, then $x^2 \equiv 0 \pmod{4}$, hence $2^n = x^2 + p \equiv p \pmod{4}$. But for $n \geq 2$, $2^n \equiv 0 \pmod{4}$, contradiction because p is odd. Thus x must be odd for $n \geq 2$.
- For $n \geq 3$, $2^n \equiv 0 \pmod{8}$. If x is odd then $x^2 \equiv 1 \pmod{8}$, so from (1)

$$p \equiv 2^n - x^2 \equiv 0 - 1 \equiv 7 \pmod{8}.$$

Hence, for $n \geq 3$, $p \equiv 7 \pmod{8}$ is necessary.

This yields a fast “congruence sieve”: among odd primes, only $p \equiv 7 \pmod{8}$ can occur for solutions with $n \geq 3$. (The small exceptional solution $x^2 + 3 = 2^2$ shows why one must explicitly separate small n .)

Search strategy for producing examples

To generate solutions of (1) for primes $p \equiv 7 \pmod{8}$, a direct constructive approach is:

1. Fix a range of exponents n (e.g., $5 \leq n \leq N$).
2. For each n , enumerate odd integers x with $1 \leq x < 2^{n/2}$.
3. Compute $p = 2^n - x^2$.
4. Keep only positive primes p .
5. Record $(p; x, n)$, noting that the same prime p may appear for multiple n and/or multiple x .

This “power-minus-square” search is transparent and aligns with how the sample solutions in Section 4 arise (many come from a fixed power such as $2^7=128$, $2^9=512$, or $2^{11} = 2048$).

A reduction for the 4^m variant

For the variant

$$x^2 + 7 = 4^m$$

Observe, $4^m = 2^{2m}$. Thus (x, m) satisfies $x^2 + 7 = 4^m$ iff $(x, 2^m)$ satisfies $x^2 + 7 = 2^n$ with n even. Therefore, uniqueness claims for the 4^m equation reduce to the known solution set of the Ramanujan–Nagell equation.

RESULT AND DISCUSSION

1 Necessary Congruence Condition (Elementary)

Proposition 4.1. Let p be an odd prime. If $x^2 + p = 2^n$ has a solution with $n \geq 3$, then x is odd and $p \equiv 7 \pmod{8}$.

Proof. As in Section 3.1.

Remark. The condition $p \equiv 7 \pmod{8}$ is not sufficient: even within $p \equiv 7 \pmod{8}$, many primes will not occur as $2^n - x^2$ for any n .

2. A clean statement for the 4^m form

Theorem 4.2. The equation $x^2 + 7 = 4^m$ has exactly one positive integer solution, namely $(x, m) = (3, 2)$.

Proof. If $x^2 + 7 = 4^m$, then $x^2 + 7 = 2^{2m}$. By the Ramanujan–Nagell theorem, the only n for which $x^2 + 7 = 2^n$ holds in natural numbers are $n \in \{3, 4, 5, 7, 15\}$, with corresponding $x \in \{1, 3, 5, 11, 181\}$.

Among these, the only even exponent is $n=4$, giving $x=3$. Hence $2m = 4 \Rightarrow m = 2$.

3. Corrected illustrative solution set for primes $p \equiv 7(\text{mod}8)$

Below is a corrected and organized list of **sample** solutions (not claimed exhaustive for each p). Each line verifies $x^2 + p = 2^n$.

- $P = 7$: $(x, n) = (1,3),(3,4),(5,5),(11,7),(181,15)$
- $P = 23$: $(x, n) = (3, 5)$ since $9+23=32$, and $(45, 11)$ since $2025 + 23 = 2048$.
- $p=31$: $(1, 5)$ since $1+31=32$; $(15, 8)$ since $225+31=256$.
- $p = 47$: $(9, 7)$ since $81 + 47 = 128$.
- $p = 71$: $(21, 9)$ since $441+71=512$.
- $p = 79$: $(7, 7)$ since $49 + 79 = 128$.
- $p = 103$: $(5, 7)$ since $25 + 103 = 128$.
- $p = 127$: $(1, 7)$ since $1+127=128$.
- $p = 151$: $(19, 9)$ since $361 + 151 = 512$.
- $p = 199$: $(43, 11)$ since $1849 + 199 = 2048$.
- $p = 223$: $(17, 9)$ since $289 + 223 = 512$.
- $p=271$: $(89, 13)$ since $7921 + 271 = 8192$.
- $p = 367$: $(41, 11)$ since $1681 + 367 = 2048$.
- $p = 431$: $(9, 9)$ since $81+431 = 512$.
- $p = 463$: $(7, 9)$ since $49 + 463 = 512$.
- $p = 487$: $(5, 9)$ since $25 + 487 = 512$.
- $p = 503$: $(3, 9)$ since $9 + 503 = 512$.

Observations.

1. Many examples cluster at fixed powers (e.g., $2^7=128$, $2^9=512$, $2^{11}=2048$), consistent with the search strategy in Section 3.2.
2. Multiplicity occurs: the same prime p may admit more than one solution (x, n) , as shown by $p = 23$ and $p = 31$.
3. Mersenne primes provide a trivial infinite "template": if $p=2^n-1$ is prime, then $(x, n) = (1, n)$ is a solution. This explains entries such as $p = 31$ (2^5-1) and $p = 127$ (2^7-1).

CONCLUSIONS

- For the prime-parameter equation $x^2 + p = 2^n$ with $n \geq 3$, the congruence condition $p \equiv 7(\text{mod}8)$ is a necessary consequence of elementary modular arithmetic.
- The condition is not sufficient; actual solvability depends on finer arithmetic structure, and primes $p \equiv 7(\text{mod}8)$ can exhibit zero, one, or multiple solutions.
- The variant $x^2 + 7 = 4^m$ reduces to the classical Ramanujan–Nagell equation and has the unique positive solution $(3, 2)$.

RECOMMENDATIONS

Recommendations for further work

1. **Fix a bound and prove completeness within that bound.** A publishable computational note should state explicit ranges (e.g., $n \leq N$, $p \leq P$) and give a complete table under those constraints.
2. **Study fixed- p completeness.** For each listed prime p , determine *all* solutions (x, n) . This typically requires deeper methods (elliptic curves, linear forms in logarithms) and should be tied to the modern Ramanujan–Nagell literature.
3. **Connect to known general frameworks.** Situate results within established theorems for $x^2 + D = A B^n$ and prime-power analogues $x^2 + D = p^z$.

FURTHER STUDY

This research still has limitations so that further research is needed on the topic of On the Exponential Diophantine Equation $x^2 + p = 2^n$ for Primes $p \equiv 7 \pmod{8}$ in order to perfect this research and increase insight for readers and writers.

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